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Contents

Articles

Reviews

Arabic Section

Abū Naṣr's Alternative Method for Determining the Solar Orbital Elements

S. Mohammad Mozaffari
Research Institute for Astronomy and Astrophysics of Maragha (RIAAM)
mozaffari@maragheh.ac.ir | s.m.muzaffari@hotmail.com
ORCID: 0000-0001-8677-8718

J. Moosavi
University of Tehran
jmoosavi@ut.ac.ir
ORCID: 0000-0002-8648-0724

Abstract: In *Almagest* III.4, Ptolemy explains the Seasons Method for determining the structural parameters of the Sun. During the Islamic period, alternative procedures were proposed to eliminate observational errors arising from observations near solstices: Using the midpoints of the seasons in the early ninth-century Mumtaḥan observations; Abū Jaʿfar al-Khāzin's recommendation to observe the Sun from longitude 120° through 240°, and from 300° to 60°; Ibrāhīm b. Sinān's method of observing the Sun at two pairs of ecliptic positions symmetric with respect to the apsidal line; and Abū al-Rayḥān al-Bīrūnī's use of two diametrically opposed longitudes with a third point near one of them, employed by Taqī al-Dīn Muḥammad b. Maʿrūf. A method devised by Abū Naṣr Manṣūr b. ʿAlī b. ʿIrāq (*ca*. 960-1036) survives in a single leaf of his *Shāhī Majisṭī* (*Royal Majisṭī*), preserved in the British Library. It uses three closely spaced points in spring or summer that intercept two arcs forming acute angles. We explain the mathematics underlying Abū Naṣr's description and estimate its theoretical deviation from the mismatch between eccentric and elliptical models. The deviations are negligible: ±0.00014 in eccentricity and +0.16° to –0.28° in the orientation of the apsidal line. We present a passage from ʿAbd al-Raḥmān al-Khāzinī's *Experimental Astronomy* indicating Khāzinī's use of Abū Naṣr's method.

Keywords: *Almagest*; Abū Naṣr ʿAbd al-Raḥmān al- Khāzinī; model of the sun; parameters; midpoints of the season.

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Resum: Al capítol III.4 de l'*Almagest*, Ptolemeu explica el Mètode de les Estacions per determinar els paràmetres estructurals del sol. Al període islàmic es van proposar procediments alternatius per eliminar errors d'observació provinents d'observacions properes als solsticis: fer servir els punts centrals de les estacions amb les observacions de l'observatori Mumtaḥan de Bagdad d'inicis del s. ix; Abū Jaʿfar al-Khāzin recomana observar el sol des de la longitud 120º fins a 240º, i des de 300º fins a 60º; el mètode d'observació del sol d'Ibrāhīm b. Sinān en dos parells de posicions eclíptiques simètriques respecte la línia de les àpsides; i l'ús per part d'Abū al-Rayḥān al-Bīrūnī de dues longituds diamètricament oposades amb un tercer punt proper a una d'elles, que també utilitza Taqī al-Dīn Muḥammad b. Maʿrūf. Ens ha arribat un mètode inventat per Abū Naṣr Manṣūr b. ʿAlī b. ʿIrāq (ca. 960-1036) en un sol foli del seu *Shāhī Majisṭī* (*Majisṭī reial*), a la British Library. Fa servir tres punts propers entre si a la tardor o a l'estiu que intercepten dos arcs que formen angles aguts. Expliquem el rerefons matemàtic de la descripció d'Abū Naṣr i fem una estimació de la desviació teòrica respecte la divergència entre els models excèntrics i el·líptics. Les desviacions són negligibles: ±0.00014 en excentricitat i entre +0.16º i –0.28º en l'orientació de la línia d'àpsides. També presentem un passatge de l'*Astronomia experimental* d''Abd al-Raḥmān al-Khāzinī que revela l'ús per part de Khāzinī's del mètode d'Abū Naṣr.

Paraules clau: *Almagest*; Abū Naṣr ʿAbd al-Raḥmān al- Khāzinī; model solar; paràmetres; punts mitjos de les estacions

1. Introduction

During the medieval period, astronomy saw the emergence of alternative methods and approaches, particularly in Islamic traditions, where they were often characterized as *ḥiyal* (singular: *ḥīla*, lit. «artifice, stratagem»; see the passage quoted from al-Khāzinī below, in Section 5). These methods were designated as mechanical (recalling that the *ʿilm al-ḥiyal* was the standard equivalent for the science of mechanics), in the sense that they rested upon heuristic reasoning.[1]

1. The underlying reasoning derives from Ptolemy's methodological guidance in *Almagest* IX.2: «to hypothesize certain/particular/some primary things, not from an apparent principle, but when they gain [our] cognition through continuous test/trial (*peira*) and fitting (*epharmoge*; Ar. *ittifāq* in Ḥajjāj's translation, LE: f. 133r:11, and *muwāfiqa muṭābiqa* in Isḥāq-Thābit's version, TN: f. 142v:23).» Mechanical reasoning has a long-standing presence in Greek mathematics, originating with Archimedes' *Method of Mechanical Theorems* and continuing in the works of Hero of

These methods were developed in response to a variety of challenging conditions. Their primary objectives were to mitigate observational errors, resolve diverse methodological difficulties—particularly in cases where standard approaches might yield unreliable outcomes and systematic inaccuracies—and simplify the lengthy, iterative procedures inherent to the methods set forth in the *Almagest*. Moreover, they aimed to rectify errors of uncertain origin that could not be adequately explained or corrected using conventional models. In essence, these approaches were fundamentally grounded in the standard techniques outlined in the *Almagest*.

Although intriguing examples have survived that illustrate the variance in outcomes of different kinematic models,[2] which captivated the attention of medieval Middle Eastern astronomers and were treated with seriousness, there is no evidence to indicate whether they considered and assessed the inevitable differences in results produced by applying various methods to the same input empirical data.

A series of such methods emerged in medieval Islamic solar astronomy for determining the Sun's orbital elements. The primary objective of this essay is to introduce one such alternative, proposed by Abū Naṣr Manṣūr b. ʿAlī b. ʿIrāq (*ca.* 960-1036), a Persian prince born probably in Gīlān on the southern shore of the Caspian Sea.[3]

Alexandria (Heath 1912, p. 13; on the *Method*, see, also, Netz 2004, pp. 2-3; Netz and Noel 2007, pp. 52-54, *et passim*; and Netz *et al*., 2011. See Lloyd 2012; Feke 2014.). Archimedes posed the mechanical method primarily as a tool for discovery—enabling certain things to be clear and further investigated—though it did not yield rigorous demonstrations. This methodological stance bears resemblance (if not identity) to Ptolemy's methodological principle articulated above. Just as mechanical reasoning in geometry provided a pathway for addressing problems where no solution could be derived through axiomatic-deductive methods, it similarly served astronomy in confronting anomalous cases—situations for which no established principles offered standard solutions. The philosophical context that establishes a direct connection between mechanical reasoning and astronomy appears to be rooted in the section related to the science of mechanics in al-Fārābī's *Enumeration of the Sciences* (al-Fārābī, ed. 1949, pp. 88-90, ed. 1996, pp. 63-64). See Goldstein 1986, p. 279-280. A more detailed treatment of this issue will be presented elsewhere (Mozaffari [forthcoming], Chapter 1).

2. E.g., Naṣīr al-Dīn al-Ṭūsī's (1201-1274) quantitative evaluation of the fitting between his own and Ptolemy's lunar models in *al-Tadhkira* II.11 (1993, Vol. 1, pp. 208-209; Hartner 1969), where he adopts Isḥāq-Thābit's term *muṭābiqa* to denote *epharmoge*.

3. See D.A. Pingree's entry «Abū Naṣr Manṣūr» in *Encyclopædia Iranica*.

One of us (SMM) previously analyzed the solar theories developed by medieval Islamic astronomers from the early ninth century to the sixteenth century. These theories are categorized into two main types (Type I and Type II) based on their dependence—or lack thereof—on Ptolemy's solar data for determining the mean motion and length of the solar year.[4] Similar studies have recently been undertaken by P. Nothaft regarding material preserved in the medieval Latin astral corpus.[5] In addition to the well-known models proposed by Abū Jaʿfar al-Khāzin[6] (900–after 970) and Ibn al-Shāṭir (d. 1375/6),[7] two further non-Ptolemaic solar models have been brought to light and examined in detail within their historical and contextual frameworks. One model, described in the cosmographical works of Quṭb al-Dīn al-Shīrāzī (d. 1311) and likely proposed by Abū Maḥmūd al-Khujandī (d. 1000), was designed to account for the decrease in solar eccentricity and the obliquity of the ecliptic.[8] The other model, seemingly inspired by a solar equatorium of possible Byzantine origin, is briefly outlined in ʿAbd al-Raḥmān al-Khāzinī's *Experimental Astronomy* (*fl.* 1100-1130), where it is employed to explain the constancy of the apparent angular size of the Sun's disk throughout the year.[9]

2. A Brief Outline of Medieval Methods for Quantifying the Ancient Solar Eccentric Model

In the *Almagest*, Ptolemy explains in detail how the structural parameters of his models for the Sun, Moon, and planets—i.e., the eccentricity, the epicycle's radius, and the orientation of the apsidal line—are determined by observations taken at three distinct points, where each celestial body occupies well-separated positions along its eccentric or epicycle. The simplest case appears in *Almagest* III.4, where the derivation of the Sun's orbital elements is elucidated through a method later known as the Seasons method. This approach is based on observations made at two consecutive equinoxes that bracket a summer solstice. The identification of

4. See Mozaffari 2013a (2025, chapter 1); 2018.
5. Nothaft 2024.
6. Samsó 1977.
7. Saliba 1987.
8. Mozaffari 2016; more details about this model will appear elsewhere.
9. Mozaffari 2022.

inconsistencies in Ptolemy's solar theory (notably, the failure to detect the motion of the solar apogee) prompted medieval astronomers to suspect that an error lay within his method. They attributed this discrepancy to his reliance on solstice observations, whose timings cannot be determined with a high degree of accuracy due to the minimal change in solar declination at those times. Consequently, alternative methods were proposed that, as much as possible, circumvent the reliance on solar observations made in the vicinity of the solstices.

In his *Ṣafāiḥ zīj* (*Astronomical tables of the plates/disks*) I.2, Abū Jaʿfar al-Khāzin proposed a general three-point method for determining the Sun's orbital elements, similar to Ptolemy's methods for deriving the lunar and planetary parameters. In this method, the Sun's longitudes are derived from its meridian altitudes measured by an armilla on three days separated at intervals of three to four months, chosen specifically when the daily variation in the Sun's declination is significant; he delineates two «safe» ecliptic zones for these measurements: one extending from the Sun's entrance to the beginning of Leo (120°) until the end of Scorpius (240°), and a second from the beginning of Aquarius (300°) to the end of Taurus (60°). Also, he advocated using the equinoxes as two of the three requisite points.[10]

A similar approach—referred to as the mid-seasons (or mid-signs) method—is based on observing the Sun at the midpoint of each of the four seasons. This method employs the same simple mathematical procedure as the Seasons method. It had been in practical use since the early ninth century, as evidenced by the Mumtaḥan observations recorded by Abū al-Rayhān al-Bīrūnī (973-1048).[11]

Abū Jaʿfar's correspondent, Ibrāhīm b. Sinān (909-946), the grandson of Thābit b. Qurra, with whom the former had shared his own model for accounting for a variable precession and the observed change in the obliquity of the ecliptic after antiquity,[12] proposed an alternative approach in his *On the Solar Motions* (completed in 933). His method required making observations of the Sun at two pairs of ecliptic positions that were symmetrical with respect to its apsidal line, and the symmetry condition was expected to fulfill by the equality of the solar longitudi-

10. al-Khāzin, *Zīj al-Ṣafāʾiḥ*, f. 27r.

11. Mozaffari 2013a (2025, chapter 1).

12. A forthcoming publication will feature a translation and a comprehensive analysis of his theory of error in solar astronomy along with his detailed account of the trepidation model as presented in *Ṣafāiḥ zīj* II.2 (ff. 29r–35v), a subject that has been frequently cited in the literature yet has never been discussed in the modern scholarship.

nal motions during the two corresponding pairs. Ibrāhīm extended this method to encompass the superior planets.[13]

In a passage written in the form of a catechism in his *al-Qānūn al-masʿūdī* VI.8,[14] Bīrūnī presented another variant of the three-point method (maybe, an elaboration of al-Khāzin's previously mentioned proposals). This variant employs two diametrically opposed points in longitude (for instance, the equinoxes) in conjunction with a third point located near one of them (such as a mid-season point). This was later quoted by Naṣīr al-Dīn al-Ṭūsī (1201-1274) as a comment on *Almagest* III.4, in his *Exposition of the* Almagest (completed in 1247).[15] Taqī al-Dīn Muḥammad b. Maʿrūf, the director of the short-lived Istanbul observatory, implemented precisely this variant with his observations of the solar noon altitudes at the equinoxes of 1579 and the third observation on 15 August 1579.[16] Similar procedures were subsequently taken up by Copernicus (1473-1543) and Tycho Brahe (1546-1601), which are well documented in the literature.[17]

A derivation of the solar orbital elements using a strictly general three-point method (i.e., one in which the observation points are not selected on any specialized basis, unlike the other variants) was carried out by Muḥyī al-Dīn al-Maghribī (d. 1283) at the Maragha observatory on the basis of his observations of the solar noon altitude on 8 September 1264 and 26 October 1264, which were first combined with an observation on 5 March 1265 and then jointly used with another observation on 25 March in the same year.[18]

3. Abū Naṣr's Variant

In his *Chronology*, Bīrūnī ambiguously speaks of a variant of the three-point method designed by his master, Abū Naṣr, and adds that he proved the superiority of this method over the other methods employed in antiquity and developed by his

13. Ibrahīm b. Sinān 1983, pp. 275-302: pp. 287-289. See Mozaffari 2013b, and its substantially updated version in 2025, chapter 6.

14. Bīrūnī, *al-Qānūn al-masʿūdī*: 1954-1956, Vol. 2, pp. 681-682.

15. Ṭūsī, *Taḥrīr al-majisṭī*, P1: pp. 98-99, P2: f. 27v, P3: ff. 44r–v, B: ff. 40r–v, AS: f. 29v.

16. Taqī al-Dīn, *Sidra*, K: f. 35r–36r, N: f. 46r–47r, V: f. 39v-40r. See Mozaffari and Steele 2015, pp. 350-354 (2025, chapter 5).

17. See Swerdlow and Neugebauer 1984, pp. 152-154; Swerdlow 2010, p. 155.

18. Saliba 1985; Mozaffari 2018b, pp. 598, 606 (2025, chapter 9).

Islamic predecessors (i.e., the aforementioned seasons and mid-seasons methods) in his (now lost) treatise titled *al-Istishhād bi-ikhtilāf al-arṣād* (*Producing witness for the difference in the observations*).[19] Abū Naṣr's method had not been previously brought to light. Intriguingly, it has survived in the only small fragment of his (again, now lost) *Shāhī Majisṭī* (*Royal Majisṭī*, an exposition of Ptolemy's *Almagest*), entitled *Istikhrāj bu'd mā bayna al-markazayn min al-Majisṭī al-Shāhī* (*Derivation of the eccentricity* [*of the Sun*, quoted] from the *Shāhī Majisṭī*), preserved on a single leaf in a codex housed in the British Library (India Office, Islamic 1270, f. 10r).[20] A transcription of the passage is provided in the Appendix. A translation follows:

> [Let] circle *AB* be the [Sun's] eccentric orb, and [let] diameter *AB* pass through the center *E* of the eccentric orb and the center *Z* of the ecliptic orb [see Figure 1]. We assume arcs *CH* and *HD* to be the two arcs [representing] the courses of the uniform movement of the Sun in the two intervals of time, which are known from the known periods and the known [Sun's] mean [daily] motion.
>
> We join *ZC*, *ZH*, and *ZD*, and prolong *ZD* to point *S* on the circumference of the [eccentric] circle. We connect *SC* and *SH*. Since angle *DZC* is known, angle *CZS* is known because it is complementary to the known angle [*DZC*], and angle *DSC* [is known] because the magnitude of arc *DC* on the perimeter of the circle is known. Thus, angle *SCZ* is known. Hence, the sides of triangle *SZC* are known [in terms of] their ratios with respect to each other. Since the ratio of *SZ* to *SH* and likewise to *SC* is known, the ratio of *SH* to *SC* is known.
>
> We join *CH*. Its ratio to the diameter is known. Also, its ratio to *SC* is known because angle *HSC* is known and the ratio of *SC* to *SH* is known. Thus, the ratio of *SC* to the diameter is known. Hence, arc *SAC* is known. Also, arc *CD* is known. So, there remains arc *DS* which is known. Also, the ratio of *SC* to *SZ* is known. So, the ratio of either *SZ* or *SD* to the diameter is known. Thus, [the resultant of] the multiplication of *SZ* and *ZD* is known. We subtract it from the square of the half-diameter [of the eccentric] **[… LACUNA? …]** the square of the perpendicular dropped from the center [*sc*. *E*] to it [i.e., *ZD*], i.e., *EL*, is known. So, *EL* and *ZL* are known, the latter being the difference between half of *SD* (known) and *SZ* (known). Thus, *EZ* [= eccentricity], which is the root of the sum of their squares [i.e., the squares of *EL* and *ZL*], is known. Therefore, the eccentricity is known.

19. al-Bīrūnī, *Chronology*, p. 184-185, En. translation, p. 167.

20. The entire codex has been digitized and is freely accessible online in the public domain: http://www.qdl.qa/en/archive/81055/vdc_100023619742.0x000001.

Also, we drop the perpendicular *DN* to the [eccentric's] diameter *AB*. Its ratio to *EL* (known) is equal to the ratio of *DZ* (known) to *EZ* (known). Thus, *DN* is known. If the point on the ecliptic corresponding to point *D* is known, the point on the ecliptic which corresponds to the apogee *A* will be known, because [the arc *AD*] between them on the eccentric orb and the eccentricity are known. Additionally, the ratio of *ZD* to *DN* is known, and angle *N* is right. So, angle *DZN* is known. This is what we wanted to prove.

The method is quite simple. Let the Sun's observed longitudes at three times (points *C*, *H*, and *D*) be denoted as λ_1, λ_2, and λ_3, respectively. The apparent motions in longitude over two intervals of time are given as follows:

(1) $\Delta\lambda_{1\to2} = \angle CZH$ and $\Delta\lambda_{2\to3} = \angle HZD$,

corresponding to the mean movements represented respectively by

(2) $\gamma_1 = \text{arc } CH = \angle CEH$ and $\gamma_2 = \text{arc } HD = \angle HED$.

In $\triangle CZS$:

(3) $\angle CZS = 180° - (\Delta\lambda_{1\to2} + \Delta\lambda_{2\to3})$,
$\angle DSC = \frac{1}{2}(\gamma_1 + \gamma_2)$, and
$\angle SCZ = (\Delta\lambda_{1\to2} + \Delta\lambda_{2\to3}) - \frac{1}{2}(\gamma_1 + \gamma_2)$.

Similarly, in $\triangle HZS$:

(4) $\angle HZS = 180° - \Delta\lambda_{2\to3}$,
$\angle DSH = \frac{1}{2}\gamma_2$, and
$\angle SHZ = \Delta\lambda_{2\to3} - \frac{1}{2}\gamma_2$.

The ratio between the sides can be easily determined using the rule of sines; e.g.,

(5) $CS/ZS = \sin(\angle CZS)/\sin(\angle SCZ)$
$HS/ZS = \sin(\angle HZS)/\sin(\angle SHZ)$

from which we can determine the ratio *HS*/*SC*.

Now, we seek a relation to express these relative ratios of the triangles' sides in terms of the eccentric's diameter or radius (R). In doing so (though our author does not explain the procedure in detail), we have:

(6) $$CH = R\ \mathrm{crd}(\gamma_1) = 2R \sin(\tfrac{1}{2}\gamma_1),$$

and

(7) $$CH^2 = CS^2 + HS^2 - 2 \cdot CS \cdot HS \cos(\tfrac{1}{2}\gamma_1),$$

in which HS can be expressed in terms of CS (from (5)). Thus, from (6) and (7), we can calculate CS, and subsequently all sides of the two triangles, including SZ (from (5)), in terms of R. Then:

(8) $$\text{arc } SAC = \mathrm{crd}^{-1}(CS/R) = 2 \sin^{-1}(CS/2R)$$

and arc DS = 360° – (arc SAC + arc CD) are known. So,

(9) $$SD = R\ \mathrm{crd}(\text{arc } DS) = 2R \sin(1/2 \text{ arc } DS)$$

is known. Knowing SZ and SD in terms of R, we can get from Euclid's *Elements* III.35:[21]

(10) $$e^2 = ZE^2 = R^2 - SZ \cdot ZD$$

The text *prima facie* seems to indicate that the right side of (10) is equal to LE^2, while it directly provides the eccentricity, unless we assume that a lacuna has occurred here in the text. Our author's presumably alternative solution involves the perpendicular drawn for the eccentric's center E to ZD, which will also be necessary shortly for the derivation of the apogee's position. EL can be calculated, similar to (10), from

(11) $$LE^2 = R^2 - (1/2SD)^2$$

21. *Elements*, pp. 63-64.